# Generalized Lorentz model description-Caputo-Fabrizio fractional derivative approach, of electrical, dielectric, conductive and magnetic processes in materials

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**Abstract**

In this study, generalized Lorentz model is basic for one-particle model in the framework of dielectric, conductive and/or magnetic responses of materials. AC conductivity studies of various $BaTiO_3$ or similar ceramics produced equivalent circuits with impedance spectra, usually within the framework of RCPE elements serial connection (CPE - constant phase element) or Cole element. This element, in the generalized Lorentz model, corresponds to Čaputo fractional derivative, who, as operator, contains a singular integral kernel in itself. However, in the literature, fractional derivatives with a non singular integral kernels have recently emerged. One of them is a Caputo-Fabrizio fractional derivative.

In this work, physical basics and all three behaviors (dielectric, conductive and magnetic) of materials and their relationships are considered in the case of electric or magnetic alternate fields, which are the tools for experimental measurements.

***Keywords—barium titanat,Caputo-Fabrizio fractional derivative, generalized Lorentz model***

**Introduction**

Basic equations of the generalized Lorentz model given in [1]. Basic set equations for The Cole model based on Caputo derivative. As is known, Caputo fractional derivative of the function $f$(t) element of $H^1$[a,b], a<b, is

$$ {}_{0}^{C}D_{dt}^{\alpha} f\left(\mathrm{t}\right) := \frac{1}{\Gamma\left(1-\alpha\right)} \cdot \int_{0+}^{t} \frac{dt'}{\left(t-t'\right)^{\alpha}} \cdot \left(\frac{df\left(\mathrm{t}\right)}{dt}\right)\Big|_{t=t'}, \ \alpha > 0, $$

Constitutive + one particle microscopic equation then (τvC , τEC -relaxation time constants for particle velocity and electric field in material, q, mq - charge and particle mass ), is

$$ \begin{aligned} &\frac{m_q}{\tau_{vC}}\left(1+\tau_{vC}^{\alpha}\cdot {}_{0}^{C}D_{dt}^{\alpha}\right) v\left(t\right) = q\left(\left(1+\tau_{EC}^{\alpha}\cdot {}_{0}^{C}D_{dt}^{\alpha}\right)\right) E\left(t\right), \\ &\left(\rho\left(0\right)+\rho\left(\infty\right)\tau_{EC}^{\alpha}\cdot {}_{0}^{C}D_{dt}^{\alpha}\right) j\left(t\right) = \left(1+\tau_{EC}^{\alpha}\cdot {}_{0}^{C}D_{dt}^{\alpha}\right) E\left(t\right), 0<\alpha\le 1, \\ &j\left(t\right) = \int_{0}^{t} dt'\cdot\sigma\left(t-t'\right) E\left(t'\right), E\left(t\right) = \int_{0}^{t} dt'\cdot\rho\left(t-t'\right) j\left(t'\right). \end{aligned} $$

Cole equation for specific impedance (resistivity) frequency dispersion in the material (it is used Laplace transform-LT), Nq is a concentration of particles (σ(ω)=1/ρ(ω)), is

$$ \rho_C\left(\omega\right) = \rho\left(\infty\right) + \frac{\rho\left(0\right)-\rho\left(\infty\right)}{1+\left(i\omega\tau_{EC}\right)^{\alpha}}, \rho\left(0\right) = \frac{m_q}{N_q q^2 \tau_{vC}}, \ \rho\left(\infty\right) = \frac{m_q}{N_q q^2 \tau_{vC}^{1-\alpha}\tau_{EC}^{\alpha}}. $$

In real experiments impedance (Z) measurement is performed .
Similar constutive equtatons for dielectric i magnetic responses of materails are

$$ \begin{aligned} &\mathbf{D}\left(t\right) + \tau_{\varepsilon CC\alpha_{\varepsilon}}^{1-\alpha_{\varepsilon}} \cdot {}_{0}^{C}D_{dt}^{1-\alpha_{\varepsilon}} \mathbf{D}\left(t\right) = \varepsilon\left(0\right)\mathbf{E}\left(t\right) + \varepsilon\left(\infty\right)\tau_{\varepsilon CC\alpha_{\varepsilon}}^{1-\alpha_{\varepsilon}} \cdot {}_{0}^{C}D_{dt}^{1-\alpha_{\varepsilon}} \mathbf{E}\left(t\right), \\ &\mathbf{H}\left(t\right) + \tau_{\mu CC\alpha_{\mu}}^{1-\alpha_{\mu}} \cdot {}_{0}^{C}D_{dt}^{1-\alpha_{\mu}} \mathbf{H}\left(t\right) = \mu\left(0\right)\mathbf{B}\left(t\right) + \mu\left(\infty\right)\tau_{\mu CC\alpha_{\mu}}^{1-\alpha_{\mu}} \cdot {}_{0}^{C}D_{dt}^{1-\alpha_{\mu}} \mathbf{B}\left(t\right). \end{aligned} $$

**Results and discussion**

In [3] motivated by recent experimental results showing that simple models are best described by fractional differential equations, tin literature he fractional Caputo derivative is applied to study of the Drude model. However, definition of the derivative has a singular kernel and cannot accurately describe the full memory effect. To correct this disadvantage, in theis work analyse d the Drude model using a CaputoFabrizio fractional derivative. Numerical simulations of these models are given in order to compare them and evaluate their effectiveness.

Caputo-Fabrizio fractional derivatives (with non-singular, non-degree integral kernels), by definition [2], for function $f$(t) - element of $H^1$[a,b], a<b (M(α)- dimension-dependent constant ),is

$$ {}^{CF}_{\ 0}D^{\alpha}_{dt} f\left(\mathrm{t}\right) := \frac{M\left(\alpha\right)}{\left(1-\alpha\right)} \cdot \int_{0+}^{t} dt' \cdot \left( \frac{df\left(\mathrm{t}\right)}{dt} \right)\Big|_{t=t'} \cdot \exp\left( \frac{-\alpha\left(t-t'\right)}{1-\alpha} \right). $$

For further work need LT formula [3]

$$ LT\left( {}^{CF}_{\ 0}D^{\alpha}_{dt} f\left(t\right) \right) = \frac{s f_{LT}\left(s\right) - f\left(0\right)}{s + \alpha\left(1-s\right)} $$

Constitutive + one particle microscopic equation then

$$ \left( \rho\left(0\right) + \rho\left(\infty\right) \, {}^{CF}_{\ 0}D^{\alpha}_{\frac{dt}{\tau_{E\alpha}}} \right) j\left(t\right) = \left( 1 + {}^{CF}_{\ 0}D^{\alpha}_{\frac{dt}{\tau_{E\alpha}}} \right) E\left(t\right), \, 0 < \alpha \le 1. $$

Impedance for this model is

$$ \rho\left(\omega\right) = \frac{\left(\rho\left(0\right) + \rho\left(\infty\right)\right) i\omega\tau_{E\alpha} + \alpha\rho\left(0\right)\left(1 - i\omega\tau_{E\alpha}\right)}{2i\omega\tau_{E\alpha} + \alpha\left(1 - i\omega\tau_{E\alpha}\right)}. $$

The standard physical explanation for impedance is: particle trajectory is fractal --Brown motion[4]. Here is, obviously, a case of another type of motion. Dielectrics and magnetics are similarly considered.
Explicitly

$$ \mathbf{D}\left(t\right) + {}^{CF}_{\ 0}D^{1-\alpha_{\varepsilon}}_{\frac{dt}{\tau_{\varepsilon CF\alpha_{\varepsilon}}}} \mathbf{D}\left(t\right) = \varepsilon\left(0\right)\mathbf{E}\left(t\right) + \varepsilon\left(\infty\right) \cdot {}^{CF}_{\ 0}D^{1-\alpha_{\varepsilon}}_{\frac{dt}{\tau_{\varepsilon CF\alpha_{\varepsilon}}}} \mathbf{E}\left(t\right), $$

$$ \mathbf{H}\left(t\right) + {}^{CF}_{\ 0}D^{1-\alpha_{\mu}}_{\frac{dt}{\tau^{1-\alpha_{\mu}}_{\mu CF\alpha_{\mu}}}} \mathbf{H}\left(t\right) = \mu\left(0\right)\mathbf{B}\left(t\right) + \mu\left(\infty\right) \, {}^{CF}_{\ 0}D^{1-\alpha_{\mu}}_{\frac{dt}{\tau_{\mu CF\alpha_{\mu}}}} \mathbf{B}\left(t\right). $$

Then

$$ \varepsilon\left(\omega\right) = \frac{\left(\varepsilon\left(0\right) + \varepsilon\left(\infty\right)\right) i\omega\tau_{E\alpha} + \alpha\varepsilon\left(0\right)\left(1 - i\omega\tau_{E\alpha}\right)}{2i\omega\tau_{E\alpha} + \alpha\left(1 - i\omega\tau_{E\alpha}\right)}, $$

$$ \mu\left(\omega\right) = \frac{\left(\mu\left(0\right) + \mu\left(\infty\right)\right) i\omega\tau_{E\alpha} + \alpha\mu\left(0\right)\left(1 - i\omega\tau_{E\alpha}\right)}{2i\omega\tau_{E\alpha} + \alpha\left(1 - i\omega\tau_{E\alpha}\right)}. $$

For complex resistivity ρ(ω) typical values for alpha are 0, 0.5 and 1. in those cases they are in order

$$\rho(\omega)=\frac{\rho(0)+\rho(\infty)}{2},$$
$$\rho(\omega)=\frac{i\omega\tau_{E0.5}\rho(\infty)+\rho(0)(1-0.5i\omega\tau_{E0.5})}{1.5i\omega\tau_{E0.5}+0.5},$$
$$\rho(\omega)=\frac{\rho(0)+\rho(\infty)i\omega\tau_{E1}}{1+i\omega\tau_{E1}}.$$

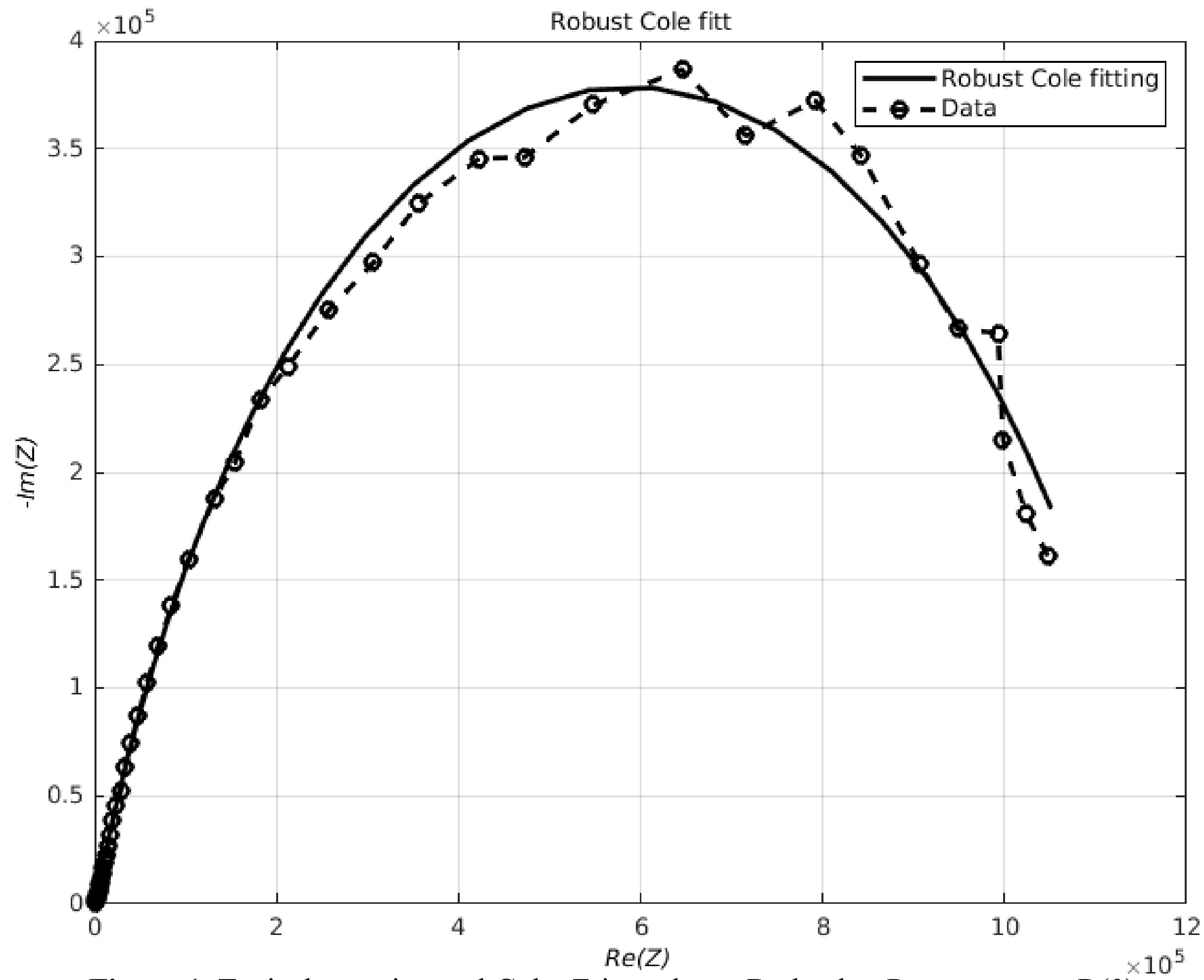


**Figure 1.** Typical experimental Cole Z-impedance Bode plot. Parametrs are: R(0)= 820; R(∞) = 1140000; τ=0.92; α = 0.766, frequencies are in interval[0.1;100000]. It is Lorentz-Caputo model.

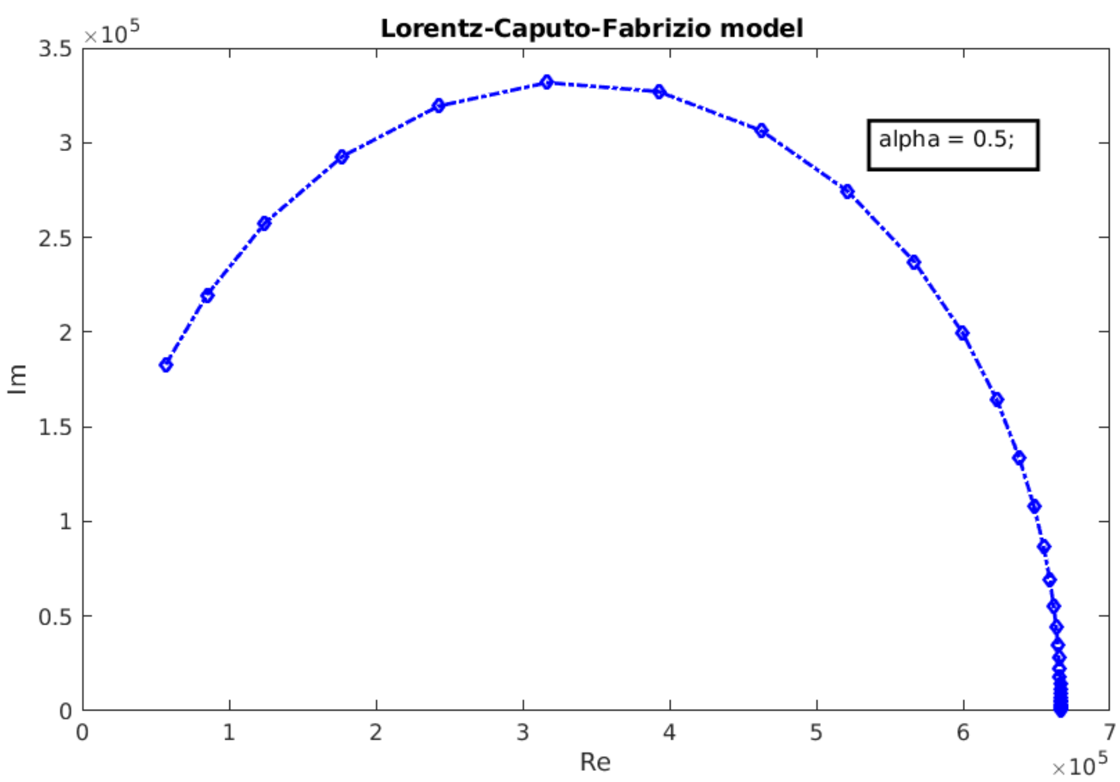


**Figure 2.**Theoretical Lorentz –Caputo-Fabrizio model Z-impedance Bode plot. Parametrs are: R(0)= 1000; R(∞) = 1000000; τ=1.0; α = 0.5; frequencies are in interval[0.1;100000].

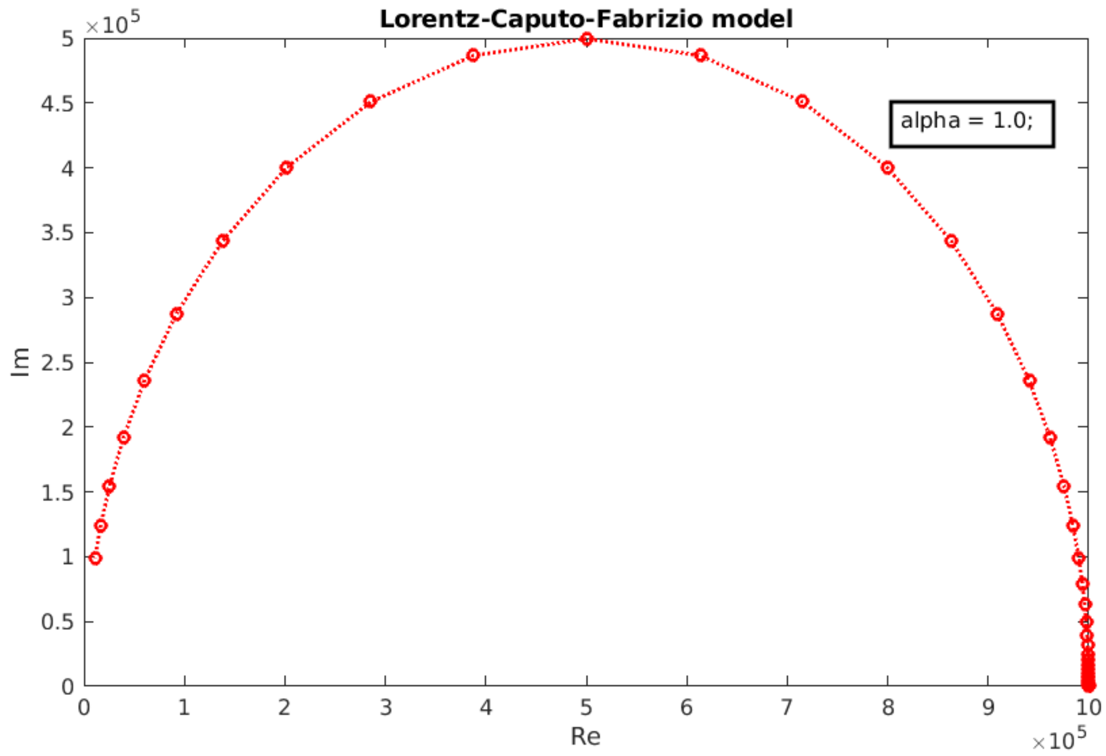


**Figure 3.**Theoretical Lorentz –Caputo-Fabrizio model Z-impedance Bode plot. Parametrs are: R(0)= 1000; R(∞) = 1000000;  τ=1.0; α =1.0; frequencies are in interval[0.1;100000].

**Conclusion**

In this case, the next interesting question arises: whether and when this models is applicable to ceramic materials, especially those based on BaTiO3 ceramics ?

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